f is *cadlag* if it is right-continuous and has left limits on its domain.

f is *decreasing* if $f(x) \geq f(y)$ whenever $x \leq y$.

f is *increasing* if $f(x) \leq f(y)$ whenever $x \leq y$.

f is *monotonic* if f is decreasing or is increasing.

f is *strictly decreasing* if $f(x) > f(y)$ whenever $x < y$.

f is *strictly increasing* if $f(x) < f(y)$ whenever $x < y$.

f is *strictly monotonic* if f is strictly decreasing or is strictly increasing.

An $\mathbb{R}$-valued operator is *positive* if it assigns nonnegative values to positive elements.

An $\mathbb{R}$-valued operator is *strictly positive* if it assigns positive values to positive elements.

The term *definite* is adjoined to the terms *positive* and *strictly positive* when viewing matrices as operators, this adjective making it clear that one is not speaking of the entries of the matrix.

ω sometimes means $\{\omega\}$.

partition is used in two distinct related ways; check the index.

A.3. Exercises on subtle distinctions

The purpose of these exercises is to focus attention on some of the conventions described in the preceding section.

Problem 1. Prove that a function $f: \mathbb{R} \to \mathbb{R}$ that is strictly increasing on an interval $[a, b]$ and on an interval $[b, c]$ is strictly increasing on the interval $[a, c]$

* **Problem 2.** Use the preceding problem and a standard calculus theorem to prove that the function $x \rightsquigarrow x - \sin x$ is strictly increasing on the interval $[-2\pi, 2\pi]$.

Problem 3. For a one-to-one function f, the notation f^{-1} has two distinct but closely related meanings. Discuss.

Problem 4. Let $a > b > 0$. Prove that the sequence $\left((a)_c^{\uparrow}/(b)_c^{\uparrow} : c = 0, 1, \dots\right)$ is strictly increasing. Does your proof show a strict increase from $c = 0$ to $c = 1$ or is a separate argument needed? Explain.

from Fristedt & Gray (1997), "A Modern Approach to Probability Theory."

Appendix B
Metric Spaces

Often, measurable sets are Borel sets, that is, members of the smallest σ-field containing all the open sets. A natural setting for Borel σ-fields is that of metric spaces, properties of which we review here. Also, some important examples will be examined.

B.1. Definition

A *metric space* consists of a set Ψ and a function $\rho: \Psi \times \Psi \to \mathbb{R}^+$ satisfying the following properties:

$$\text{(i) } \rho(x, y) = 0 \text{ if and only if } y = x\,;$$
$$\text{(ii) } \rho(x, y) = \rho(y, x)\,;$$
$$\text{(iii) } \rho(x, z) \leq \rho(x, y) + \rho(y, z)\,.$$

The last two properties are called *symmetry* and *triangle inequality*, respectively. The function ρ is called the *metric* on Ψ and its value at a particular pair (x, y) is the *distance* between x and y. A metric space, thus defined, is denoted by (Ψ, ρ), or more briefly by Ψ if there is no ambiguity concerning the metric.

For x a member of a metric space (Ψ, ρ) and $\varepsilon > 0$, the sets

$$\{y\colon \rho(x, y) < \varepsilon\}\,, \quad \{y\colon \rho(x, y) \leq \varepsilon\}\,, \quad \text{and} \quad \{y\colon \rho(x, y) = \varepsilon\}$$

are called the *open ball, closed ball,* and *sphere* of radius ε centered at x. A subset B of a metric space with metric ρ is an *open set* if for every $x \in B$ there exists $\varepsilon > 0$ such that the open ball of radius ε centered at x is a subset of B. It is easy to use the triangle inequality to prove that every open ball is an open set. A set is a *closed set* if its complement is open. It is easy to prove that all spheres and closed balls are closed sets.

For a set $C \subseteq \Psi$, the *interior* of C is the largest open subset of C; the *closure* of C is its smallest closed superset; and the *boundary* of C, denoted by ∂C, consists of those points in its closure that are not also in its interior. It is possible to prove the existence of the interior and closure of any subset of a metric space.

A subset B of a set C in a metric space is *dense in C* if every ball centered at a point in C contains a member of B. The modifying phrase "in C" is often omitted if C is the entire metric space. A metric space is *separable* if it contains a countable dense subset.

A set in a metric space is *bounded* if it is contained in some ball. It is *totally bounded* if for every $\varepsilon > 0$, it is contained in the union of a finite collection of balls of radius less than ε. It is *compact* if every open cover of it has a finite subcovering. (A collection of sets is called a *cover* of a set C if C is a subset of their union; the cover is *open* if each of the sets in the cover is open.) A set is *relatively compact* if it has compact closure. If any of the adjectives *bounded, totally bounded,* and *compact* apply to the set of all points in a metric space, then the corresponding adjective is also used for the metric space itself.

* **Problem 1.** Prove the following facts about sets in any metric space.

- Finite intersections and arbitrary unions of open sets are open.
- Finite unions and arbitrary intersections of closed sets are closed.
- Finite unions and arbitrary intersections of compact sets are compact.
- Every compact set is closed.
- A closed subset of a compact set is compact.
- The intersection of a collection of compact sets is empty if and only if the intersection of some finite subcollection is empty.
- Any set in a metric space is itself a metric space with the inherited metric.

Problem 2. Use the first two items in the preceding problem to prove the facts mentioned above: every set has an interior, possibly empty; every set has a closure.

B.2. Sequences

A sequence $(x_1, x_2, \dots)$ in a metric space (Ψ, ρ) *converges* to a point $x \in \Psi$ if, for every $\varepsilon > 0$, there exists an integer p such that $\rho(x_n, x) < \varepsilon$ whenever $n \geq p$. The sequence is *Cauchy* if, for every $\varepsilon > 0$, there exists p such $\rho(x_m, x_n) < \varepsilon$ whenever $n \geq m \geq p$.

Proposition 1. *Every convergent sequence in a metric space is Cauchy, and every Cauchy sequence which has a convergent subsequence converges.*

Problem 3. Prove the preceding proposition.

A metric space in which every Cauchy sequence is convergent is said to be *complete.*

Proposition 2. *Every sequence in a totally bounded metric space has a subsequence that is Cauchy.*

Problem 4. Prove the preceding proposition.

A set C in a metric space is *relatively sequentially compact* if every sequence in C has a subsequence that converges. In case the subsequence can always be chosen so that its limit belongs to C, C is *sequentially compact.* The proof of the following result will be omitted.

Proposition 3. *Compactness is equivalent to sequential compactness; relative compactness is equivalent to relative sequential compactness.*

In practice one often proves that a sequence converges by simultaneously proving relative sequential compactness and that every convergent subsequence has the same limit. The next proposition entails a recipe for doing this.

Proposition 4. *A sequence $(x_n : n = 1, 2, \dots)$ converges to a limit y if and only if every subsequence of $(x_n : n = 1, 2, \dots)$ has a further subsequence that converges to y.*

* **Problem 5.** Prove the preceding proposition.

B.3. Continuous functions

A function g from one metric space (Ψ_1, ρ_1) to another metric space (Ψ_2, ρ_2) is *continuous at* $x \in \Psi_1$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho_2(g(y), g(x)) < \varepsilon$ for every y satisfying $\rho_1(x, y) < \delta$. Equivalently, g is continuous at x if for every sequence $(x_n : n = 1, 2, \dots)$ in Ψ_1 converging to x, it is true that $g(x_n) \to g(x)$ as $n \to \infty$. The function g is *continuous* if it is continuous at each point. It is *uniformly continuous* if δ can be chosen to depend only on ε and g, but not on x.

Problem 6. Prove that if $f : \Psi_1 \to \Psi_2$ is a function from one metric space to another, then f is continuous if and only if for any open set $A \subset \Psi_2$, $f^{-1}(A)$ is an open subset of Ψ_1.

Problem 7. Prove that if f is a continuous function from one metric space to another, then the image under f of any compact set is compact. *Hint:* Use Problem 6.

B.4. Important metric spaces

The function $(x, y) \rightsquigarrow |y - x|$ is a metric for $\mathbb{R}$. Another metric for $\mathbb{R}$ is $(x, y) \rightsquigarrow |\arctan y - \arctan x|$. These two metrics for $\mathbb{R}$ make $\mathbb{R}$ into two different metric spaces. With the first metric, $\mathbb{R}$ is complete but not bounded, and with the second it is totally bounded but not complete. However, the open sets determined by the two metrics are easily seen to be identical. Thus these two metrics turn $\mathbb{R}$ into the same measurable space.

The metric $(x, y) \rightsquigarrow |\arctan y - \arctan x|$ for $\mathbb{R}$ can be extended to a metric for $\overline{\mathbb{R}}$ by defining $\arctan\infty = \pi/2$ and $\arctan(-\infty) = -\pi/2$. With this metric large finite real numbers are 'close' to ∞ and negative finite real numbers of large absolute value are 'close' to $-\infty$. The metric space $\overline{\mathbb{R}}$ is complete and compact. The function $x \rightsquigarrow \arctan x$ from $\overline{\mathbb{R}}$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is a continuous bijection with a continuous inverse, where the metric for $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is $(u, v) \rightsquigarrow |u - v|$.

In $\mathbb{R}^d$ we let $|x|$ denote the Euclidean distance from the point $x \in \mathbb{R}^d$ to the origin. The function $(x, y) \rightsquigarrow |x - y|$ is a metric for $\mathbb{R}^d$ which turns $\mathbb{R}^d$ into a complete metric space.

The standard way of making $\mathbf{C}[0, 1]$, the set of continuous functions on the interval $[0, 1]$, into a metric space is to define the distance between f and g to equal $\max\{|f(t) - g(t)| : 0 \leq t \leq 1\}$.

Problem 8. Show that the closed ball of radius 1 centered at the 0 function in $\mathbf{C}[0, 1]$ is not totally bounded. *Hint:* Construct an infinite sequence $(f_n : n = 1, 2, \dots)$ in B such that the distance between f_m and f_n equals 2 for $m \neq n$.

Theorem 5. [Arzelà-Ascoli] *A subset A of $\mathbf{C}[0, 1]$ is relatively sequentially compact if and only if $\{f(0) : f \in A\}$ is a bounded set of real numbers and, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ and $f \in A$.*

We omit the proof of this theorem. Notice that δ in it is not permitted to depend on f. A set A of functions is said to be *equicontinuous* at a point x if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ whenever $|y - x| < \delta$ and $f \in A$. If δ can be chosen to be independent of x, then the family A is said to be *uniformly equicontinuous*. Thus, the Arzelà-Ascoli Theorem can be stated as: a subset of $\mathbf{C}[0, 1]$ is relatively sequentially compact if and only if it is uniformly equicontinuous and the set of its values at 0 is bounded. Moreover, 'uniformly' need not be mentioned because it is a consequence of equicontinuity at each point and the fact that $[0, 1]$ is compact. [In fact, some people use 'equicontinuous' to mean 'uniformly equicontinuous'.]

APPENDIX C
Topological Spaces

Metric spaces, described in the Appendix B, are examples of a more general structure that will be described in this appendix.

C.1. Concepts

A *topological space* is a pair $(\Omega, \mathcal{O})$ where Ω is a set and $\mathcal{O}$ a family of subsets of Ω satisfying the following properties:

(i) $\mathcal{O}$ is closed under arbitrary unions;
(ii) $\mathcal{O}$ is closed under finite intersections;
(iii) $\emptyset \in \mathcal{O}$;
(iv) $\Omega \in \mathcal{O}$.

The collection $\mathcal{O}$ is called a *topology* on Ω. (Properties (iii) and (iv) are redundant in view of the standard convention that the union and intersection of an empty collection of sets are the empty set and universal set, respectively. The members of $\mathcal{O}$ are said to be *open* and their complements are *closed*. (This use of 'closed' should not be confused with its use to describe an operation as in (i) and (ii) above.) Often one refers to a topological space by mentioning the universal set Ω, rather than both Ω and the topology $\mathcal{O}$. When doing this care is required, since it is possible, as illustrated by Problem 18, for two different topological spaces having the same universal set to appear in the same discussion.

It is easily shown that a set C in a topological space has a largest open subset, which is its *interior*, and a smallest closed subset, which is its *closure*. The *boundary* of C, denoted by ∂C, consists of those points in its closure that are not also in its interior.

Problem 1. The solution of some problem in Appendix B contains a proof that every metric space is a topological space. Which problem is that?

Problem 2. Why is it true that, with one exception, every topological space has at least two sets having the property of being both open and closed? What is the one exception?

Problem 3. Prove that the collection $\mathcal{C}$ of closed sets in a topological space has the following two properties:

(i) $\mathcal{C}$ is closed under arbitrary intersections;

(ii) $\mathcal{C}$ is closed under finite unions.

Problem 4. Prove that every set in a topological space has both an interior and a closure.

A *neighborhood* of a point in a topological space is any set that contains some open set of which the point is a member. (Some people place an additional condition on a set for it to be a neighborhood of a point—namely, that it itself be open.) A topological space is *Hausdorff* if for any two points x and w in the space there exist neighborhoods of x and w that have empty intersection. Hausdorff spaces almost always suffice for applications to probability.

* **Problem 5.** Prove that a point x belongs to the boundary of a set B if and only if every neighborhood of x contains at least one point in B and at least one point in B^c.

* **Problem 6.** Prove that the boundary of a set is also the boundary of its complement.

Problem 7. Prove that every metric space is Hausdorff.

A subset of a topological space Ω is *compact* if every open cover of it has a finite subcovering. It is *relatively compact* if its closure is compact. In case Ω itself is compact, the topological space is a *compact space.* The next two results describe connections between compactness and closedness.

Proposition 1. *A closed subset of a compact set in a topological space is a compact subset in that topological space.*

Problem 8. Prove the preceding proposition.

Proposition 2. *Every compact set in a Hausdorff space is closed.*

PROOF. For a proof by contradiction suppose that B is a compact set that is not closed. Let $x \in \partial B \setminus B$ and let w be any member of B. Since the topological space is Hausdorff, there exist neighborhoods of x and w that have empty intersection, neighborhoods that with no loss of generality we may take to be open. The complement of the open neighborhood of w is a closed neighborhood of x. Therefore, the collection

$$\{N^c : N \text{ a closed neighborhood of } x\}$$

is an open covering of B. Because B is compact this covering contains a a finite subcovering, say

$$\{N_1^c, N_2^c, \dots, N_k^c\}.$$

Let O_i denote the interior of N_i, $1 \leq i \leq k$. No point in $O = \cap_{i=1}^k O_i$ is covered by the finite subcovering, but O, being the intersection of a finite number of open neighborhoods of x, is itself an open neighborhood of x —and, because $x \in \partial B$, O contains a member of B by Problem 5, a member that is not covered by the finite subcovering. Therefore, we have arrived at the desired contradiction. □

The following problem introduces a topological space that cannot be viewed as a metric space, no matter how one chooses to specify a metric.

* **Problem 9.** Let $\mathcal{O}$ consist of all subsets O of $\mathbb{R}$ having the property that for every $x \in O$ there exists $\varepsilon > 0$ such that the interval $[x, x+\varepsilon) \subseteq O$. Prove $(\mathbb{R}, \mathcal{O})$ is a topological space. For this topology, decide which intervals are open, which are closed, and which are compact.

C.2. Compactification

Sometimes there are strong reasons for working with a compact topological space even when the topological space of interest is not compact.

Let $(\Omega, \mathcal{O})$ be a topological space and adjoin to Ω an additional point—call it ∞ —to obtain a set $\Omega^* = \Omega \cup \{\infty\}$. Let $\mathcal{O}^*$ consist of all members of $\mathcal{O}$ and all subsets of Ω^* whose complements in Ω^* are compact subsets of Ω.

* **Problem 10.** Prove that $(\Omega^*, \mathcal{O}^*)$, as defined in the preceding paragraph, is a compact space.

The topological space $(\Omega^*, \mathcal{O}^*)$ introduced above is called the *one-point compactification* of the topological space $(\Omega, \mathcal{O})$.

Example 1. The one-point compactification of the real line $\mathbb{R}$ with the usual topology has the effect of 'putting' negative numbers of large absolute value and large positive numbers 'close' to the same member, ∞, of the compactification.

A more commonly used compactification of $\mathbb{R}$ is its *two-point compactification* $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. The open sets in $\overline{\mathbb{R}}$ are the open sets of $\mathbb{R}$, sets of the form $[-\infty, x)$ for some $x \in \mathbb{R}$, sets of the form $(x, \infty]$ for some $x \in \mathbb{R}$, and unions of sets of these types.

Problem 11. Prove that $\overline{\mathbb{R}}$ as just described is a topological space. Also, show that this topological space is the one induced by the metric $(x, y) \rightsquigarrow |\arctan x - \arctan y|$ which was introduced in the last section of Appendix B.

C.3. Product topologies

Let J be an arbitrary index set. For $j \in J$, let $(\Omega_j, \mathcal{O}_j)$ be a topological space. Set

$$\Omega = \prod_{j \in J} \Omega_j ,$$

$$\mathcal{N} = \left\{ \prod_{j \in J} O_j : O_j \text{ open in } \mathcal{O}_j \text{ and } O_j = \Omega_j \text{ for all but finitely many } j \right\},$$

and $\mathcal{O}$ equal to the collection of all unions of members of $\mathcal{N}$.

Problem 12. Prove that $(\Omega, \mathcal{O})$ as just described is a topological space.

The collection $\mathcal{O}$ defined above is called the *product topology* of the topologies $\mathcal{O}_j$, and the topological space $(\Omega, \mathcal{O})$ is the *product* of the topological spaces $(\Omega_j, \mathcal{O}_j)$.

We omit the proof of the following important theorem about product spaces.

Theorem 3. [Tychonoff] *A product of compact spaces is a compact space.*

Example 2. By the preceding theorem $\overline{\mathbb{R}}^\infty = \prod_{k=1}^\infty \overline{\mathbb{R}}$ is compact; that is, the set of all sequences in $\overline{\mathbb{R}}$ is compact.

C.4. Relative topology

For $(\Omega, \mathcal{O})$ a topological space and $\Psi \subseteq \Omega$, let

$$\mathcal{P} = \{O \cap \Psi : O \in \mathcal{O}\} .$$

Problem 13. Prove that $(\Psi, \mathcal{P})$ as just defined is a topological space.

The topological space $(\Psi, \mathcal{P})$ described above is called a *topological subspace* of $(\Omega, \mathcal{O})$ and $\mathcal{P}$ is the *relative topology* on Ψ. The next exercise shows that sets can change their topological character when a topology is replaced by its relative topology, but the subsequent proposition shows that the compactness property is stable under such a replacement.

* **Problem 14.** Give an example that shows that a set that is not a member of a topology $\mathcal{O}$ may be open in a relative topology induced by $\mathcal{O}$ on a set Ψ. Prove, however, that this phenomenon cannot happen if $\Psi \in \mathcal{O}$.

Proposition 4. *Let $(\Omega, \mathcal{O})$ be a topological space and $\Psi \subseteq \Omega$. Then a subset C of Ψ is compact with respect to the topology $\mathcal{O}$ if and only if it is compact with respect to the relative topology induced by $\mathcal{O}$ on Ψ.*

Problem 15. Prove the preceding proposition.

C.5. Limits and continuous functions

In Definition 5 and Definition 8 we essentially copy appropriate versions of definitions that are standard for the topological space $\mathbb{R}$.

Definition 5. Let f be a function from a topological space Υ to a topological space Ω. The function f is said to be *continuous at a point* $y \in \Upsilon$ if $f^{-1}(N)$ is a neighborhood of y for every neighborhood N of $f(y)$. And f is *continuous* if it is continuous at each point in Υ.

Proposition 6. *A function from one topological space to another is continuous if and only if the inverse image of every open set is open.*

Proof. Let $f : \Upsilon \to \Omega$. For one direction suppose that the inverse image under f every open set in Ω is open in Υ, and consider an arbitrary $y \in \Upsilon$ and an arbitrary neighborhood N of $f(y)$. There exists an open set $O \subseteq N$ for which $f(y) \in O$. Then $f^{-1}(O)$ contains y, is open, and is a subset of $f^{-1}(N)$. Therefore, $f^{-1}(N)$ is a neighborhood of y. Since y is arbitrary, f is continuous.

For the other direction, suppose that f is continuous and consider an arbitrary open set O in Ω. Since O is a neighborhood of each of its members, $f^{-1}(O)$ is a neighborhood of all of its members and thus, for each y in $f^{-1}(O)$, there exists an open set N_y such that $y \in N_y \subseteq f^{-1}(O)$. Therefore $f^{-1}(O) = \bigcup_{y \in f^{-1}(O)} N_y$, which being the union of open sets is open. □

Proposition 7. *Let $f : \Upsilon \to \Omega$ be continuous and suppose that Υ is compact. Then the image of f is compact.*

Problem 16. Prove the preceding theorem.

Problem 17. Use relative topology to adapt the preceding discussion to the case where the domain of f is a subset of Υ.

Problem 18. Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f is right-continuous (as usually defined) if and only if it is continuous when the domain has the topology of Problem 9 and the target has the usual topology.

Definition 8. Let f be a function with target a topological space Ω and domain a subset of a topological space Υ. Let $y \in \Upsilon$ and suppose that every neighborhood of y contains a point different from y in the domain of f. We say that

$$\lim_{z \to y} f(z) = x$$

if for every neighborhood N of x there is a neighborhood M of y such that $f(z) \in N$ whenever $z \neq y$ is in the intersection of M and the domain of f.

Problem 19. Suppose that $f\colon \Upsilon \to \Omega$ is continuous. Prove that

$$\lim_{z\to y} f(z) = f(y)$$

for every $y \in \Upsilon$ for which the one-point set $\{y\}$ is not open.

Example 3. Make $\mathbb{Z}^+$ into a topological space by calling every subset open, and let $\overline{\mathbb{Z}}^+$ denote the one-point compactification of $\mathbb{Z}^+$. The compact sets in $\mathbb{Z}^+$ are the finite sets, so that the neighborhoods of ∞ in $\overline{\mathbb{Z}}^+$ are those sets that contain ∞ and have finite complements. Consider an arbitrary function $f\colon \mathbb{Z}^+ \to \Omega$, where Ω is any topological space. From the preceding discussion it follows that

$$\lim_{z\to\infty} f(z) = x$$

if and only if for every neighborhood N of x there is a member m of $\mathbb{Z}^+$ such that $f(z) \in N$ whenever $z > m$. Hence, we see that sequential convergence is encompassed by Definition 8.

Comment: Another way to view the topology on $\overline{\mathbb{Z}}^+$ is that it is the relative topology induced by the usual topology on $\overline{\mathbb{R}}$.

Proposition 9. *Suppose that a sequence $(x_n\colon n = 1, 2, \dots)$ of points in a closed set C in a topological space Ω converges to a point $x \in \Omega$. Then $x \in C$.*

Problem 20. Prove the preceding proposition.

Theorem 10. *Any sequence in a compact Hausdorff space has a convergent subsequence.*

Problem 21. Prove the preceding theorem.

Problem 22. Let $(\Omega, \mathcal{O})$ be the product of topological spaces $(\Omega_j, \mathcal{O}_j)$, $j = 1, 2, \dots$. For each $n = 1, 2, \dots$, let

$$\omega_n = (\omega_{n,1}, \omega_{n,2}, \omega_{n,3}, \dots)$$

be a point in Ω. Show that the sequence $(\omega_n\colon n = 1, 2, \dots)$ converges in Ω if and only if the sequence $(\omega_{n,j}\colon n = 1, 2, \dots)$ converges in Ω_j for each fixed j.

Problem 23. In the topological space of Problem 9 find an infinite sequence that does not converge even though it would converge were the topology the usual topology for $\mathbb{R}$.

APPENDIX D
Riemann-Stieltjes Integration

The Riemann integral $\int_a^b f(x)\,dx$ is, by definition, the limit of sums of the form

$$\sum_{j=1}^{n} f(\xi_j)\,(x_j - x_{j-1}),$$

where $a = x_0 < x_1 < \cdots < x_n = b$ and $\xi_j \in [x_{j-1}, x_j]$ for each j. In this appendix we replace the differences $x_j - x_{j-1}$ by $g(x_j) - g(x_{j-1})$ for some function g. This procedure leads to a type of integral that lies somewhere between the Riemann integral and the Lebesgue integral in generality. One advantage that this integral has over the Lebesgue integral is that it satisfies an integration by parts formula that can be quite useful for calculational purposes.

D.1. The Riemann-Stieltjes integral

The basic setting consists of two functions f and g defined on a closed bounded interval $[a, b]$ of the real line. By a *point partition* of the interval $[a, b]$ we mean a finite subset of $[a, b]$ containing both a and b. (In many books a point partition is identified by the one-word term 'partition', which we use to denote a partition of a set.) We typically write the members of a point partition in increasing order. Thus, when we say that $\{x_0, x_1, \dots, x_n\}$ is a point partition of $[a, b]$, it is to be understood that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

To emphasize this point we may use the contrived notation $\{a = x_0 < x_1, \cdots < x_n = b\}$, possibly omitting a and b from the notation if the interval on which the point partition is based is clear from context. The *mesh* of the point partition $\{a = x_0 < x_1, \cdots < x_n = b\}$ is the maximum of the numbers $x_j - x_{j-1}$, $1 \le j \le n$.

A point partition of the interval $[a, b]$ is said to be a *refinement* of a second point partition if it contains the second point partition as a subset.

A *Riemann-Stieltjes sum* of f with respect to g corresponding to a point partition $\{x_0 < x_1 < \cdots < x_n\}$ of $[a, b]$ is a sum

$$\sum_{j=1}^{n} f(\xi_j)\,[g(x_j) - g(x_{j-1})]\,,$$

where $\xi_j \in [x_{j-1}, x_j]$ for each j. Since each ξ_j is only constrained to lie in a certain interval, there are typically many Riemann-Stieltjes sums corresponding to a particular point partition.

Definition 1. The function f is *Riemann-Stieltjes integrable* with respect to the function g on the interval $[a, b]$ if there is some number γ such that for every $\varepsilon > 0$, there is a point partition P of $[a, b]$ for which the difference between γ and any Riemann-Stieltjes sum of f with respect to g corresponding to any refinement of P has absolute value less than ε. In case there is such a γ, the *Riemann-Stieltjes integral* of f with respect to g on the interval $[a, b]$ is said to exist and equal γ, and one writes

$$\gamma = \int_a^b f\,dg = \int_a^b f(x)\,dg(x)\,,$$

either suppressing the independent variable x or writing it explicitly.

Suppose that g is an increasing function. Any Riemann-Stieltjes sum for a given point partition is bounded above by the *upper Riemann-Stieltjes sum*

$$\sum_{j=1}^{n} \sup\{f(x) \colon x_{j-1} \le x \le x_j\}\,[g(x_j) - g(x_{j-1})]\,,$$

for that point partition and below by the *lower Riemann-Stieltjes sum*, obtained by replacing 'sup' by 'inf'. It is easy to see that f is Riemann-Stieltjes integrable with respect to g if and only if for every $\varepsilon > 0$ there is a point partition for which the corresponding upper and lower Riemann-Stieltjes sums are finite and differ by less than ε.

* **Problem 1.** Calculate $\int_{1/2}^{4} x^2\,d\lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer that is no larger than x.

* **Problem 2.** Let

$$g(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 3^{-n} & \text{if } 2^{-n} \le x < 2^{-(n-1)}\,,\ n \in \mathbb{Z}^+ \setminus \{0\} \\ 1 & \text{if } x \ge 1\,. \end{cases}$$

Evaluate $\int_{-7}^{5} (1 - x^2)\,dg(x)$.

Problem 3. Verify the following equalities:

$$\int_0^1 x\,dx^2 = \tfrac{2}{3}\,;$$
$$\int_0^1 2F(x)\,dF(x) = F^2(1) - F^2(0) \quad \text{for } F \text{ continuous and increasing.}$$

Problem 4. Let

$$F(x) = \begin{cases} x/3 & \text{if } 0 \le x < 1 \\ 1/2 & \text{if } 1 \le x < 2 \\ x/3 & \text{if } 2 \le x \le 3\,. \end{cases}$$

Prove that $\int_0^3 F(x)\,dF(x)$ does not exist.

Problem 5. Prove that on any closed bounded interval, every continuous functio is Riemann-Stieltjes integrable with respect to every function that is the differenc of two monotone functions. *Hint:* Use the uniform continuity of the continuou function.

Problem 6. Let f and g be monotone functions on an interval $[a, b]$ and suppos that f is left-continuous and g is right-continuous. Prove that f is Riemanr Stieltjes integrable with respect to g.

It is important to notice that there are no differentiability assumptions i Problem 5 or Problem 6.

D.2. Relation to the Riemann integral

The following proposition shows how to change some Riemann-Stieltjes integral into Riemann integrals which can then often be evaluated by using the Funda mental Theorem of Calculus.

Proposition 2. *Let g be a function with a continuous first derivative on a interval $[a, b]$ and f a bounded $\mathbb{R}$-valued function on $[a, b]$. Then fg' is Rieman integrable on $[a, b]$, if and only if f is Riemann-Stieltjes integrable with respec to g on $[a, b]$ in which case*

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)g'(x)\,dx\,. \tag{D.1}$$

PROOF. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a point partition of $[a, b$ and let $\xi_j \in [x_{j-1}, x_j]$ for $1 \le j \le n$. The corresponding Riemann sum of fg' i

$$\sum_{j=1}^{n} f(\xi_j)g'(\xi_j)\,(x_j - x_{j-1})\,,$$

and the corresponding Riemann-Stieltjes sum of f with respect to g is

$$\sum_{j=1}^{n} f(\xi_j)\,[g(x_j) - g(x_{j-1})]\,,$$

By the Mean-Value Theorem, there exist numbers $\eta_j \in [x_{j-1}, x_j]$ such that this Riemann-Stieltjes sum equals

$$\sum_{j=1}^{n} f(\xi_j) g'(\eta_j)\,(x_j - x_{j-1})\,.$$

We conclude that the absolute value of the difference between the Riemann sum of fg' and the Riemann-Stieltjes sum of f with respect to g is bounded by

$$\sum_{j=1}^{n} \big|f(\xi_j)\big| \cdot \big|g'(\xi_j) - g'(\eta_j)\big|\,(x_j - x_{j-1})\,. \tag{D.2}$$

The quantity (D.2) is bounded by the product of three numbers: any bound s of $|f|$, $(b-a)$, and the maximum of $|g'(v) - g'(u)|$ taken over $u, v \in [x_{j-1}, x_j]$, $1 \le j \le n$. The third of these factors can be made arbitrarily small by taking the mesh of P to be sufficiently small, say less than some ε. For all refinements of such a P there is a correspondence between Riemann sums of fg' and Riemann-Stieltjes sums of f with respect to g such that corresponding sums differ by less than $s(b-a)\varepsilon$. The desired conclusion follows. □

Problem 7. Discuss how Proposition 2 might be of use in treating an integral $\int_a^b f\,dg$ even if g does not satisfy all the conditions in that proposition.

Problem 8. Evaluate $\int_{-1}^{2} |x-1|\,d|x|$ by using Proposition 2, the Fundamental Theorem of Calculus, and your response for Problem 7. Do this problem by breaking the integral into no more than two pieces for the application of the Fundamental Theorem.

D.3. Change of variables

The formula for making the same change of variables in both functions of a Riemann-Stieltjes integral is easy to remember.

Proposition 3. *Let φ be a strictly increasing continuous function on an interval $[a, b]$. Then*

$$\int_a^b (f \circ \varphi)\,d(g \circ \varphi) = \int_{\varphi(a)}^{\varphi(b)} f\,dg\,,$$

in the sense that if either side exists then so does the other and they are equal.

Problem 9. Prove the preceding proposition.

Problem 10. Without concerning yourself with appropriate hypotheses, show how the preceding proposition is related to the usual change of variables formula for Riemann integrals.

D.4. Integration by parts

The following theorem, which gives a general integration by parts formula, is the main reason for the existence of this appendix.

Theorem 4. *Suppose that a function f is Riemann-Stieltjes integrable with respect to a function g on an interval $[a, b]$. Then g is Riemann-Stieltjes integrable with respect to f on $[a, b]$ and*

$$\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - \int_a^b g\,df\,.$$

Proof. Set $\gamma = \int_a^b f\,dg$. Let $\varepsilon > 0$ and choose a point partition P such that for every point partition $\{x_0, x_1, \ldots, x_n\}$ that is a refinement of P and every choice of $\xi_j \in [x_{j-1}, x_j]$,

$$\Big|\sum_{j=1}^{n} f(\xi_j)\,[g(x_j) - g(x_{j-1})] - \gamma\Big| < \varepsilon\,.$$

For such a point partition consider an arbitrary Riemann-Stieltjes sum for g with respect to f:

$$\sum_{j=1}^{n} g(\eta_j)\,[f(x_j) - f(x_{j-1})]\,.$$

We set $\eta_0 = a$ and $\eta_{n+1} = b$ in order to rewrite this Riemann-Stieltjes sum as

$$\begin{aligned} &f(b)g(b) - f(a)g(a) - \sum_{i=1}^{n+1} f(x_{i-1})\,[g(\eta_i) - g(\eta_{i-1})] \\ &= f(b)g(b) - f(a)g(a) - \sum_{i=2}^{n+1} f(x_{i-1})\,[g(x_{i-1}) - g(\eta_{i-1})] \\ &\qquad - \sum_{i=1}^{n} f(x_{i-1})\,[g(\eta_i) - g(x_{i-1})]\,. \end{aligned} \tag{D.3}$$

The combination of these last two summations is the negative of a Riemann-Stieltjes sum of f with respect to g for the point partition

$$P' = \{a = x_0 = \eta_0 \le \eta_1 \le x_1 \le \eta_2 \le \cdots \le \eta_n \le x_n = \eta_{n+1} = b\}\,,$$

the possibility of equality in this description of P' causing no problem, but only indicating that there may be less than $2n$ subintervals determined by P'. Since P' is a refinement of P, (D.3) differs from $f(b)g(b) - f(a)g(a) - \gamma$ by less than ε. Since ε is arbitrary the proof is complete. □

It is worth noticing that the integration by parts formula is symmetric in f and g. It is also worth observing that the above proof uses a simple technique called *summation by parts* that has a slightly messy appearance in the proof because of the interlacing of two sequences. Here is the simple useful formula isolated by itself.

Proposition 5. *Let $(a_0, a_1, \ldots, a_n)$ and $(b_0, b_1, \ldots, b_n)$ be two finite sequences of real numbers. Then*

$$\sum_{j=1}^{n} a_j \left[b_j - b_{j-1} \right] = a_n b_n - a_0 b_0 - \sum_{i=1}^{n} b_{i-1} \left[a_i - a_{i-1} \right].$$

Problem 11. Convince yourself that the preceding proposition is true.

Problem 12. Redo Problem 1 by using integration by parts.

Problem 13. Show that all monotone functions on an interval are Riemann-Stieltjes integrable with respect to every continuous function on that interval (even a continuous function whose derivative exists nowhere). *Hint:* Use Problem 5 and an important theorem.

* **Problem 14.** Let f be an $\mathbb{R}$-valued function on an interval $[a, b]$, and suppose that for every $x \in [a, b]$,

$$f(x+) = \lim_{y \searrow x} f(y) \quad \text{and} \quad f(x-) = \lim_{y \nearrow x} f(y)$$

both exist as members of $\mathbb{R}$. Show that if g has a continuous derivative on $[a, b]$, then f and g are Riemann-Stieltjes integrable with respect to each other on $[a, b]$.

D.5. Improper Riemann-Stieltjes integrals

The treatment of improper Riemann-Stieltjes integrals parallels that of improper Riemann integrals. In particular, when one is using various theorems about Riemann-Stieltjes integrals, such as integration by parts, it is wise to first write a given improper Riemann-Stieltjes integral as a limit of proper Riemann-Stieltjes integrals, then use the theorems, and finally pass to the limit.

Problem 15. Replace the interval $[-7, 5]$ of integration in Problem 2 by the interval $(-\infty, \infty)$ and then do the problem created by this replacement.

Problem 16. Does the improper integral

$$\int_0^\infty \frac{(-1)^{\lceil x \rceil}}{x} \, d\lfloor x \rfloor$$

exist as a finite number? (Here $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer larger than or equal to x and the largest integer less than or equal to x, respectively.) Give attention to the issue of existence of appropriate proper Riemann-Stieltjes integrals.

Appendix E

Taylor Approximations, ℂ-Valued Logarithms

For some portions of this book it is important to have a definition of $\log \circ \beta$ for a complex-valued function β. Under some restrictions, such a definition is presented in the Section 2. A second theme appears throughout this appendix—that of approximating or bounding transcendental functions by polynomials.

E.1. Some inequalities based on the Taylor formula

From the Taylor formula with remainder one easily gets the following families of inequalities:

$$\begin{aligned} 1 - \frac{v^2}{2!} &\le \cos v \le 1, & v &\in \mathbb{R}, \\ 1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \frac{v^6}{6!} &\le \cos v \le 1 - \frac{v^2}{2!} + \frac{v^4}{4!}, & v &\in \mathbb{R}, \\ &\vdots \quad \vdots \quad \vdots; \end{aligned}$$

$$\begin{aligned} v - \frac{v^3}{3!} &\le \sin v \le v, & v &\in \mathbb{R}^+, \\ v - \frac{v^3}{3!} + \frac{v^5}{5!} - \frac{v^7}{7!} &\le \sin v \le v - \frac{v^3}{3!} + \frac{v^5}{5!}, & v &\in \mathbb{R}^+, \\ &\vdots \quad \vdots \quad \vdots; \end{aligned}$$

$$\begin{aligned} 1 - \frac{x^1}{1!} &\le e^{-x} \le 1, & x &\in \mathbb{R}^+, \\ 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} &\le e^{-x} \le 1 - x + \frac{x^2}{2!}, & x &\in \mathbb{R}^+, \\ &\vdots \quad \vdots \quad \vdots; \end{aligned}$$