AMS 261 — Probability Theory (Spring 2005)
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- Characterization of Borel subsets of $\overline{\mathbb{R}}$
- Borel subsets of $\overline{\mathbb{R}}^+$

Borel subsets of $\overline{\mathbb{R}}$

Consider the topological (and metric) space $(\mathbb{R}, \mathcal{O})$, with $\mathcal{O}$ the collection of open sets in $\mathbb{R}$ and $\mathcal{B} = \sigma(\mathcal{O})$ the Borel σ-field (in $\mathbb{R}$).

Also consider $(\overline{\mathbb{R}}, \overline{\mathcal{O}})$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is the two-point compactification of $\mathbb{R}$ and $\overline{\mathcal{O}}$ is the collection of open sets in $\overline{\mathbb{R}}$, consisting of the open sets in $\mathbb{R}$, sets of the form $[-\infty, x)$ for some $x \in \mathbb{R}$, sets of the form $(x, \infty]$ for some $x \in \mathbb{R}$, and unions of sets of these types. Then, $\overline{\mathcal{B}} = \sigma(\overline{\mathcal{O}})$ is the Borel σ-field (in $\overline{\mathbb{R}}$).

In order to characterize the sets in $\mathcal{B}$ through sets in $\overline{\mathcal{B}}$ we'll need the following important result:

Lemma: Let $\mathcal{C}$ be a class of subsets of some sample space Ω, $A \subset \Omega$ and denote by $\mathcal{C} \cap A$ the class $\{B \cap A : B \in \mathcal{C}\}$. Also let $\sigma(\mathcal{C})$ be the σ-field generated by $\mathcal{C}$. Then $\sigma_A(\mathcal{C} \cap A) = \sigma(\mathcal{C}) \cap A$ (where on the left hand side A rather than Ω is regarded as the entire space).

Proof

First of all we must show that $\sigma(\mathscr{C}) \cap A = \{G \cap A : G \in \sigma(\mathscr{C})\}$ is a σ-field of subsets of A.

$\sigma(\mathscr{C}) \cap A$ obviously forms a collection of subsets of A.

- Since $\Omega \in \sigma(\mathscr{C}) \Rightarrow \Omega \cap A = A \in \sigma(\mathscr{C}) \cap A$ (first axiom)
- Consider $D \in \sigma(\mathscr{C}) \cap A \Rightarrow D = G \cap A$, $G \in \sigma(\mathscr{C})$.

The complement of D with respect to A is

$$D^c = A \sim (G \cap A) = A \cap (G \cap A)^c \quad \text{(now the complement is w.r.t. } \Omega\text{)}$$
$$= A \cap (G^c \cup A^c)$$
$$= (A \cap G^c) \cup (A \cap A^c) \quad (A \cap A^c = \phi)$$
$$= G^c \cap A$$

Since $G \in \sigma(\mathscr{C}) \Rightarrow G^c \in \sigma(\mathscr{C}) \Rightarrow (G^c \cap A) \in \sigma(\mathscr{C}) \cap A$
$\Rightarrow D^c \in \sigma(\mathscr{C}) \cap A$. (second axiom)

- Finally consider $D_1, D_2, \ldots \in \sigma(\mathscr{C}) \cap A$
$\Rightarrow D_i = G_i \cap A$, with $G_i \in \sigma(\mathscr{C})$; $i = 1, 2, \ldots$

Then $\bigcup_{i=1}^{\infty} D_i = \bigcup_{i=1}^{\infty} (G_i \cap A)$

$$= \left(\bigcup_{i=1}^{\infty} G_i\right) \cap A \in \sigma(\mathscr{C}) \cap A$$

since $\bigcup_{i=1}^{\infty} G_i \in \sigma(\mathscr{C})$, $\sigma(\mathscr{C})$ being a σ-field. (third axiom).

We have $\mathscr{C} \subseteq \sigma(\mathscr{C}) \Rightarrow \mathscr{C} \cap A \subseteq \sigma(\mathscr{C}) \cap A$ (both collections of subsets of A)
$\Rightarrow \sigma_A(\mathscr{C} \cap A) \subseteq \sigma_A(\sigma(\mathscr{C}) \cap A) = \sigma(\mathscr{C}) \cap A$
since we have seen that $\sigma(\mathscr{C}) \cap A$ is a σ-field of subsets of A.

Thus it remains to show that $\sigma(\mathscr{C}) \cap A \subseteq \sigma_A(\mathscr{C} \cap A)$ to establish the result. In other words we must prove that $G \cap A \in \sigma_A(\mathscr{C} \cap A)$, for any $G \in \sigma(\mathscr{C})$. Following the "good sets" idea define $\mathcal{H} = \{G \in \sigma(\mathscr{C}) / G \cap A \in \sigma_A(\mathscr{C} \cap A)\} \subseteq \sigma(\mathscr{C})$

$\mathcal{H}$ is a collection of subsets of Ω. We want to show that $\mathcal{H}$ is a σ-field.

- $A = \Omega \cap A \in \sigma_A(\mathcal{C} \cap A)$, since $\sigma_A(\mathcal{C} \cap A)$ is a σ-field of subsets of A, so $\Omega \in \mathcal{H}$. (first axiom)
- Consider $G \in \mathcal{H} \Rightarrow G \cap A \in \sigma_A(\mathcal{C} \cap A)$

$$\Rightarrow (G \cap A)^c \in \sigma_A(\mathcal{C} \cap A) \qquad \text{(complement w.r.t. } A\text{)}$$

$$\Rightarrow A \setminus (G \cap A) = A \cap (G \cap A)^c \qquad (\quad " \quad " \quad \Omega)$$

$$= A \cap (G^c \cup A^c)$$

$$= (A \cap G^c) \cup (A \cap A^c)$$

$$= G^c \cap A \in \sigma_A(\mathcal{C} \cap A)$$

$$\Rightarrow G^c \in \mathcal{H}. \qquad \text{(second axiom)}$$

- Finally consider $G_1, G_2, \dots \in \mathcal{H} \Rightarrow G_i \cap A \in \sigma_A(\mathcal{C} \cap A)$ for $i = 1, 2, \dots$

Then $\bigcup_{i=1}^{\infty}(G_i \cap A) \in \sigma_A(\mathcal{C} \cap A)$

$$\Rightarrow \left(\bigcup_{i=1}^{\infty} G_i\right) \cap A \in \sigma_A(\mathcal{C} \cap A) \Rightarrow \bigcup_{i=1}^{\infty} G_i \in \mathcal{H}. \text{ (third axiom)}$$

Now if $B \in \mathcal{C} \Rightarrow B \cap A \in \mathcal{C} \cap A \subseteq \sigma_A(\mathcal{C} \cap A)$

$$\Rightarrow B \in \mathcal{H}$$

So $\mathcal{C} \subseteq \mathcal{H} \Rightarrow \sigma(\mathcal{C}) \subseteq \sigma(\mathcal{H}) = \mathcal{H}$ (since $\mathcal{H}$ is a σ-field) and since from the definition $\mathcal{H} \subseteq \sigma(\mathcal{C})$, we have that $\mathcal{H} = \sigma(\mathcal{C})$, hence for any $G \in \sigma(\mathcal{C})$, $G \cap A \in \sigma_A(\mathcal{C} \cap A)$, which completes the proof. ∎

Now we will apply the lemma with $\Omega = \overline{\mathbb{R}}$, $\mathcal{C} = \overline{\mathcal{O}}$ and $A = \mathbb{R}$. From the form of the open sets in $\overline{\mathbb{R}}$, we have that

$$\overline{\mathcal{O}} \cap \mathbb{R} = \{\overline{O} \cap \mathbb{R} : \overline{O} \in \overline{\mathcal{O}}\} = \mathcal{O}.$$

Hence $\sigma_{\mathbb{R}}(\overline{\mathcal{O}} \cap \mathbb{R}) = \sigma(\overline{\mathcal{O}}) \cap \mathbb{R} \Rightarrow \sigma_{\mathbb{R}}(\mathcal{O}) = \overline{\mathbb{B}} \cap \mathbb{R}$

$$\Rightarrow \quad \mathbb{B} = \overline{\mathbb{B}} \cap \mathbb{R} = \{ \overline{B} \cap \mathbb{R} : \overline{B} \in \overline{\mathbb{B}} \}.$$

Thus a Borel set B in $\mathbb{R}$ is equal to the intersection of a Borel set $\overline{B}$ in $\overline{\mathbb{R}}$ and $\mathbb{R}$.

Now since $\overline{B} \subseteq \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, we have

$$\begin{aligned}
\overline{B} &= (\overline{B} \setminus \mathbb{R}) \cup (\overline{B} \cap \mathbb{R}) \\
&= (\overline{B} \cap \mathbb{R}^c) \cup B \qquad \text{(complement w.r.t. } \overline{\mathbb{R}}) \\
&= (\overline{B} \cap \{-\infty, \infty\}) \cup B
\end{aligned}$$

The intersection $\overline{B} \cap \{-\infty, \infty\}$ can be:
ϕ if $\overline{B} \subset \mathbb{R}$, or $\{-\infty\}$ if, for example, $\overline{B} = [-\infty, x)$, for some $x \in \mathbb{R}$, or $\{\infty\}$ if, for example, $\overline{B} = (x, \infty]$, for some $x \in \mathbb{R}$ or finally $\{-\infty, \infty\}$ if, for example, $\overline{B} = [-\infty, x) \cup (y, \infty]$, for $x, y \in \mathbb{R}$.

Hence, in conclusion, a Borel subset of $\overline{\mathbb{R}}$ is characterized as the union of a Borel subset of $\mathbb{R}$ and one of the four subsets of $\{-\infty, \infty\}$.

Borel subsets of $\overline{\mathbb{R}}^+$

Topological space $(\overline{\mathbb{R}}^+, \overline{\mathcal{O}}^+)$, where $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}$ and $\overline{\mathcal{O}}^+$ the collection of open sets in $\overline{\mathbb{R}}^+$ consisting of the open sets in $\mathbb{R}^+$ (i.e., sets of the form (a, b), $a > 0$, $[0, b)$ and unions of such), sets of the form $(a, \infty]$, for some $a > 0$ and unions of sets of these types.

The Borel σ-field is $\overline{\mathbb{B}}^+ = \sigma(\overline{\mathcal{O}}^+)$

Now apply the lemma with $\Omega = \overline{\mathbb{R}}^+$, $\mathcal{C} = \overline{\mathcal{O}}^+$ and $A = \mathbb{R}^+$

We have $\overline{\mathcal{O}}^+ \cap \mathbb{R}^+ = \{ \overline{O}^+ \cap \mathbb{R}^+ : \overline{O}^+ \in \overline{\mathcal{O}}^+ \} = \mathcal{O}^+$ the collection of open sets in $\mathbb{R}^+$. Then $\sigma_{\mathbb{R}^+}(\overline{\mathcal{O}}^+ \cap \mathbb{R}^+) = \sigma(\overline{\mathcal{O}}^+) \cap \mathbb{R}^+$

$$\Rightarrow \sigma_{\mathbb{R}^+}(\mathcal{O}^+) = \overline{\mathcal{B}}^+ \cap \mathbb{R}^+ \Rightarrow \mathcal{B}^+ = \overline{\mathcal{B}}^+ \cap \mathbb{R}^+$$
$$= \{ \overline{B}^+ \cap \mathbb{R}^+ : \overline{B}^+ \in \overline{\mathcal{B}}^+ \}$$

with $\mathcal{B}^+$ the Borel σ-field of $\mathbb{R}^+$ which can be described as the collection of the Borel subsets of $\mathbb{R}$ that contain only nonnegative numbers. To see this we can apply again the lemma with $\Omega = \mathbb{R}$, $\mathcal{C} = \mathcal{O}$ and $A = \mathbb{R}^+$ (now $\mathcal{O} \cap \mathbb{R}^+ = \mathcal{O}^+$) to get

$$\mathcal{B}^+ = \mathcal{B} \cap \mathbb{R}^+ = \{ B \cap \mathbb{R}^+ : B \in \mathcal{B} \}.$$

Returning to the relation $\mathcal{B}^+ = \overline{\mathcal{B}}^+ \cap \mathbb{R}^+$, we can write a Borel subset $\overline{B}^+$ of $\overline{\mathbb{R}}^+$ in the form

$$\begin{aligned}
\overline{B}^+ &= (\overline{B}^+ \setminus \mathbb{R}^+) \cup (\overline{B}^+ \cap \mathbb{R}^+) \\
&= (\overline{B}^+ \cap (\mathbb{R}^+)^c) \cup B^+ \qquad \text{(complement w.r.t. } \overline{\mathbb{R}}^+\text{)} \\
&= (\overline{B}^+ \cap \{\infty\}) \cup B^+ \\
&= \begin{cases} \phi \cup B^+ = B^+ \\ B^+ \cup \{\infty\} \end{cases}
\end{aligned}$$

Hence a Borel subset of $\overline{\mathbb{R}}^+$ is either equal to a Borel subset of $\mathbb{R}^+$ or to the union of a Borel subset of $\mathbb{R}^+$ and $\{\infty\}$.