AMS 261: Probability Theory (Fall 2017)

Homework 1 (due Tuesday 10/17)

1. Consider a sample space Ω.

(a) Prove that any intersection of σ-fields (of subsets of Ω) is a σ-field. That is, if $\mathcal{F}_j$, $j \in J$, are σ-fields on Ω (with J an arbitrary index set, countable or uncountable), then show that $\mathcal{F} = \bigcap_{j \in J} \mathcal{F}_j$ is a σ-field.

(b) Show by counterexample that a union of σ-fields may not be a σ-field.

2. Given a sample space Ω and a collection $\mathcal{E}$ of subsets of Ω, the σ-field generated by $\mathcal{E}$, $\sigma(\mathcal{E})$, is defined as the intersection of all σ-fields on Ω that contain $\mathcal{E}$. (As discussed in class, $\sigma(\mathcal{E})$ is the smallest σ-field that contains $\mathcal{E}$.)

(a) Consider two collections $\mathcal{E}_1$ and $\mathcal{E}_2$ of subsets of Ω. Show that if $\mathcal{E}_1 \subseteq \mathcal{E}_2$, then $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$.

(b) As in part (a), let $\mathcal{E}_1$ and $\mathcal{E}_2$ be collections of subsets of the sample space Ω. Prove that if $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ and $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$, then $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$.

3. Let $\mathcal{F}$ be a collection of subsets of a sample space Ω.

(a) Suppose that $\Omega \in \mathcal{F}$, and that when $A, B \in \mathcal{F}$ then $A \cap B^c \in \mathcal{F}$. Show that $\mathcal{F}$ is a field.

(b) Suppose that $\Omega \in \mathcal{F}$, and that $\mathcal{F}$ is closed under the formation of complements and finite pairwise disjoint unions. Show by counterexample that $\mathcal{F}$ need not be a field.

4. Consider the sample space $\Omega = (0, 1]$ and the collection $\mathcal{B}_0$ of all finite pairwise disjoint unions of subintervals of $(0, 1]$. That is, any member B of $\mathcal{B}_0$ is of the form $B = \bigcup_{i=1}^{n} (a_i, b_i]$, where n is finite, and for each $i = 1, ..., n$, $0 \leq a_i < b_i \leq 1$, with $(a_i, b_i] \cap (a_j, b_j] = \emptyset$ for any $i \neq j$.
Show that $\mathcal{B}_0$ augmented by the empty set is a field, but not a σ-field.

5. Let $\Omega = \{\omega_1, \omega_2, ...\}$ be a countable set, $\{p_n : n = 1, 2, ...\}$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} p_n = 1$, and $\mathcal{F}$ be the collection of all subsets of Ω. For each $A \in \mathcal{F}$, define the set function

$$P(A) = \sum_{\{n:\, \omega_n \in A\}} p_n.$$

Show that $(\Omega, \mathcal{F}, P)$ is a probability space.