AMS 261: Probability Theory (Fall 2017)

Homework 2 (due Thursday 10/26)

1. Let $\{A_n : n = 1, 2, ...\}$ be a countable sequence of subsets of a sample space Ω.
(a) Assume that $\{A_n : n = 1, 2, ...\}$ is an increasing sequence, that is, $A_n \subseteq A_{n+1}$, for all $n \geq 1$. Show that $\lim_{n\to\infty} A_n$ exists, and $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$.
(b) Assume that $\{A_n : n = 1, 2, ...\}$ is a decreasing sequence, that is, $A_{n+1} \subseteq A_n$, for all $n \geq 1$. Show that $\lim_{n\to\infty} A_n$ exists, and $\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$.

2. Consider countable sequences, $\{A_n : n = 1, 2, ...\}$, $\{B_n : n = 1, 2, ...\}$ and $\{C_n : n = 1, 2, ...\}$, of subsets of the same sample space Ω. Assume that $A_n \subseteq B_n \subseteq C_n$, for all $n \geq K$ for some sufficiently large positive integer K. Moreover, suppose that $\limsup_{n\to\infty} C_n \subseteq \liminf_{n\to\infty} A_n$. Prove that each of $\lim_{n\to\infty} A_n$, $\lim_{n\to\infty} B_n$ and $\lim_{n\to\infty} C_n$ exists, and that all three limits are the same.

3. Consider a measurable space $(\Omega, \mathcal{F})$ and a set function P: $\mathcal{F} \longrightarrow [0,1]$, which satisfies $P(\Omega) = 1$, and $P(A \cup B) = P(A) + P(B)$ for any A and B in $\mathcal{F}$ with $A \cap B = \emptyset$. Moreover, assume that P is continuous, that is, $P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n)$, for any sequence $\{A_n : n = 1, 2, ...\}$ of sets in $\mathcal{F}$ for which $\lim_{n\to\infty} A_n$ exists. Prove that P is a probability measure on $(\Omega, \mathcal{F})$.

4. Prove that any non-decreasing function from $\mathbb{R}$ to $\mathbb{R}$ is measurable. (Assume the usual Borel σ-field on $\mathbb{R}$.)

5. Let $(\Omega_j, \mathcal{F}_j)$, $j = 1, 2, 3$, be measurable spaces. Consider measurable functions $X : \Omega_1 \to \Omega_2$ and $Y : \Omega_2 \to \Omega_3$, and define the composition function $Y \circ X : \Omega_1 \to \Omega_3$ by $Y \circ X(\omega_1) = Y(X(\omega_1))$, for any $\omega_1 \in \Omega_1$. Show that $Y \circ X$ is a measurable function.

6. Consider a sequence $\{X_n : n = 1, 2, ...\}$ of $\mathbb{R}$-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Let C be the set of $\omega \in \Omega$ such that $\{X_n(\omega) : n = 1, 2, ...\}$ is a convergent numerical sequence. Prove that $C \in \mathcal{F}$.

7. Let X and Y be $\mathbb{R}$-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$, and consider the subset of Ω defined by $A = \{\omega \in \Omega : X(\omega) \neq Y(\omega)\}$.
(a) Prove that A is an event in $\mathcal{F}$.
(**Hint**: Recall the *Archimedean Property* of the real numbers, according to which, for any two real numbers a and b with $a < b$, there exists a rational number q such that $a < q < b$.)
(b) Assume that $P(A) = 0$. Prove that $P(X^{-1}(B)) = P(Y^{-1}(B))$ for any Borel subset B of $\mathbb{R}$ (in which case, we say that the distributions of X and Y are equal).