AMS 261: Probability Theory (Fall 2017)

Homework 3 (due Tuesday 11/14)

1. Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\mathbb{R}^+$–valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Assume that all random variables X_n have the same distribution, with distribution function given by $F(x) = 1 - \exp(-x)$, $x \in \mathbb{R}^+$.
   • Show that
   $$P\left(\liminf_{n\to\infty} \{\omega \in \Omega : X_n(\omega) \leq (1+\delta)\log(n)\}\right) = 1,$$
   for any fixed $\delta > 0$.

2. Let F and G be distribution functions on $\mathbb{R}$ such that $G(t) \leq F(t)$, for all $t \in \mathbb{R}$ (in which case, G is said to be *stochastically larger* than F).
   • Construct two $\mathbb{R}$-valued random variables X and Y, defined on the same probability space $(\Omega, \mathcal{F}, P)$, such that the distribution function of X is G, the distribution function of Y is F, and $P(X \geq Y) = 1$.

3. Consider a simple random variable X defined on some probability space $(\Omega, \mathcal{F}, P)$, and let F be its distribution function. Denote by $F(x^-) = \lim_{y \nearrow x} F(y)$ (or equivalently, $F(x^-) = \lim_{n\to\infty} F(x_n)$ for an arbitrary increasing sequence $\{x_n : n = 1, 2, ...\}$ converging to x).
   • Show that the expectation of X can be written in the form
   $$\mathrm{E}(X) = \sum_{x \in \mathbb{R}} x\{F(x) - F(x^-)\}.$$

4. Let X be a simple random variable (taking both negative and positive values) defined on some probability space $(\Omega, \mathcal{F}, P)$.
   • Show that expectation definitions 1 (for simple random variables) and 3 (for general random variables taking values on the extended real line) are equivalent.

5. Consider an $\overline{\mathbb{R}}^+$-valued random variable X defined on some probability space $(\Omega, \mathcal{F}, P)$, and assume that $\mathrm{E}(X) < \infty$. Let $A = \{\omega \in \Omega : X(\omega) = +\infty\}$, and note that, based on the general definition for $\overline{\mathbb{R}}^+$-valued measurable functions, we have $A \in \mathcal{F}$.
   • Show that X is almost surely finite, that is, $P(A) = 0$.

6. Consider a sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}^+$-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that the sequence is (pointwise) increasing, that is, for all n and for each $\omega \in \Omega$, $X_n(\omega) \leq X_{n+1}(\omega)$. Denote by X the pointwise limit of $\{X_n : n = 1, 2, ...\}$, that is, for each $\omega \in \Omega$, $X(\omega) = \lim_{n\to\infty} X_n(\omega)$, and assume that $\mathrm{E}(X) < \infty$. Define the *variance* for X by $\mathrm{Var}(X) = \mathrm{E}\{(X - \mathrm{E}(X))^2\}$, and similarly, for each n, $\mathrm{Var}(X_n) = \mathrm{E}\{(X_n - \mathrm{E}(X_n))^2\}$. (In general, the variance for a random variable Y with finite expectation $\mathrm{E}(Y)$ is given by $\mathrm{Var}(Y) = \mathrm{E}\{(Y - \mathrm{E}(Y))^2\}$, whether finite or infinite.)
   • Prove that $\mathrm{Var}(X) = \lim_{n\to\infty} \mathrm{Var}(X_n)$.