AMS 261: Probability Theory (Fall 2017)

Homework 5 (due Wednesday 12/13)

1. Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a measurable space $(\Psi, \mathcal{G})$, where $\mathcal{G}$ is the σ-field on space Ψ. Consider the collection $\mathcal{A}$ of subsets of Ω consisting of $X^{-1}(B)$ for all $B \in \mathcal{G}$.
   • Show that $\mathcal{A}$ is a σ-field on Ω.

2. For $k = 1, 2, ...$, consider random variables $X_k : (\Omega, \mathcal{F}, P) \to (\Psi_k, \mathcal{G}_k)$ and measurable functions $\varphi_k : (\Psi_k, \mathcal{G}_k) \to (\Theta_k, \mathcal{H}_k)$. Assume that the countable sequence of random variables $\{X_k : k = 1, 2, ...\}$ is independent.
   • Prove that the sequence $\{\varphi_k \circ X_k : k = 1, 2, ...\}$ is independent.

3. Let $\{A_n : n = 1, 2, ...\}$ be a countable independent sequence of events on a probability space $(\Omega, \mathcal{F}, P)$.
   • Prove that $P(\bigcap_{n=1}^{\infty} A_n) = \prod_{n=1}^{\infty} P(A_n)$.
   (**Note:** For a countable sequence of reals, $\{b_n : n = 1, 2, ...\}$, the infinite product $\prod_{n=1}^{\infty} b_n$ is defined by $\lim_{n\to\infty} \prod_{k=1}^{n} b_k$, provided this limit exists.)

4. Consider two countable sequences of events, $\{A_n : n = 1, 2, ...\}$ and $\{B_n : n = 1, 2, ...\}$, on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that, for each n, A_n and B_n are independent. Moreover, assume that $A = \lim_{n\to\infty} A_n$ and $B = \lim_{n\to\infty} B_n$ exist.
   • Show that A and B are independent.

5. A sequence $\{X_n : n = 1, 2, ...\}$ of $\mathbb{R}$-valued random variables, defined on a common probability space $(\Omega, \mathcal{F}, P)$, is said to converge completely if for any $k = 1, 2, ...$, $\sum_{n=1}^{\infty} P(|X_n| > k^{-1}) < \infty$.
   • Show that if $\{X_n : n = 1, 2, ...\}$ converges completely, then $\lim_{n\to\infty} X_n = 0$ almost surely.

6. Construct a sequence $\{X_n : n = 1, 2, ...\}$ of $\mathbb{R}^+$-valued random variables (i.e., $X_n \geq 0$, for all n) that satisfies $\sum_{n=1}^{\infty} P(X_n > k^{-1}) < \infty$, for any $k = 1, 2, ...$, but for which $\lim_{n\to\infty} \mathrm{E}(X_n) \neq 0$.

7. Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Assume that each random variable X_n is uniformly distributed on $(0, 1)$, hence, $P(c < X_n < d) \equiv P(\{\omega \in \Omega : X_n(\omega) \in (c, d)\}) = d - c$, for any $0 \leq c < d \leq 1$.
   • Show that the sequence $\{1/(n^2 X_n) : n = 1, 2, ...\}$ converges almost surely to 0 as $n \to \infty$.