AMS 261: Probability Theory (Fall 2017)

Random infinite series

Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of **independent** $\mathbb{R}$–valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. A key theoretical question for the associated infinite series, $\sum_{n=1}^{\infty} X_n$, involves study of conditions for its almost sure (a.s.) convergence.

By definition, $\sum_{n=1}^{\infty} X_n$ converges a.s. if the sequence of random variables $\{S_n : n = 1, 2, ...\}$, where $S_n = \sum_{i=1}^{n} X_i$, converges a.s. Hence, the definition exploits the relation between series of random variables and series of reals (note that for each $\omega \in \Omega$, $\sum_{n=1}^{\infty} X_n(\omega)$ is a series of reals). In fact, it can be shown that for independent $\{X_n : n = 1, 2, ...\}$, $\sum_{n=1}^{\infty} X_n$ either converges a.s. or diverges a.s., with some key convergence results including:

Theorem 1: For a sequence $\{X_n : n = 1, 2, ...\}$ of independent $\mathbb{R}$–valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$, $\sum_{n=1}^{\infty} X_n$ converges a.s. to an $\mathbb{R}$–valued random variable Z if and only if $\sum_{n=1}^{\infty} X_n$ converges in probability to Z.

Theorem 2: Consider a sequence $\{X_n : n = 1, 2, ...\}$ of independent $\mathbb{R}$–valued random variables, defined on a common probability space $(\Omega, \mathcal{F}, P)$. Assume that each X_n has finite variance and that $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty$. Then, $\sum_{n=1}^{\infty} \{X_n - \mathrm{E}(X_n)\}$ converges a.s.

Kolmogorov three-series theorem: Let $\{X_n : n = 1, 2, ...\}$ be an independent sequence of $\mathbb{R}$–valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. For each n, consider the truncated version of X_n defined by $Y_n = X_n 1_{(|X_n| \leq b)}$, where b is a positive real constant. Then $\sum_{n=1}^{\infty} X_n$ converges a.s. if and only if each of the following three series converges: $\sum_{n=1}^{\infty} \mathrm{E}(Y_n)$; $\sum_{n=1}^{\infty} \mathrm{Var}(Y_n)$; and $\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(|X_n| > b)$.

Weak and strong laws of large numbers

Laws of large numbers involve convergence results for functionals of $S_n = \sum_{i=1}^{n} X_i$ (the average, $n^{-1} \sum_{i=1}^{n} X_i$, being a standard example), where the sequence of random variables $\{X_n : n = 1, 2, ...\}$ is independent. Various versions of laws of large numbers exist, but the key results include the "Weak law of large numbers" (WLLN) (yielding convergence in probability) and the "Strong law of large numbers" (SLLN) (resulting in almost sure convergence). In particular, the two standard versions for the SLLN correspond to different assumptions for the sequence $\{X_n : n = 1, 2, ...\}$.

Weak law of large numbers

Consider an independent sequence of $\mathbb{R}$–valued random variables $\{X_n : n = 1, 2, ...\}$, defined on a common probability space $(\Omega, \mathcal{F}, P)$, and let $S_n = \sum_{i=1}^{n} X_i$. Assume that for each n, $\mathrm{E}(X_n^2) < \infty$, and that $\lim_{n\to\infty} b_n^{-2} \sum_{i=1}^{n} \mathrm{Var}(X_i) = 0$, where $\{b_n : n = 1, 2, ...\}$ is a sequence of reals. Then, $b_n^{-1}(S_n - \mathrm{E}(S_n))$ converges to 0 in probability.

Proof. Application of Chebyshev's inequality to random variable $b_n^{-1} S_n$.

Strong law of large numbers

Consider an independent sequence of $\mathbb{R}$–valued random variables $\{X_n : n = 1, 2, ...\}$, defined on a common probability space $(\Omega, \mathcal{F}, P)$, and let $S_n = \sum_{i=1}^{n} X_i$. Moreover, let $\{b_n : n = 1, 2, ...\}$ be an increasing sequence of positive reals such that $\lim_{n\to\infty} b_n = \infty$. Assume that for each n, $\mathrm{E}(X_n) = 0$ and $\mathrm{E}(X_n^2) < \infty$, and that $\sum_{i=1}^{\infty} b_i^{-2} \mathrm{E}(X_i^2) < \infty$. Then, $b_n^{-1} S_n$ converges to 0 almost surely.

Proof. Based on Theorem 2 and a result from series of real numbers, the Kronecker lemma.
Kronecker lemma: Consider a sequence $\{x_n : n = 1, 2, ...\}$ of reals such that $\sum_{i=1}^{\infty} x_i < \infty$, and another sequence $\{b_n : n = 1, 2, ...\}$ of positive reals which is increasing, with $\lim_{n\to\infty} b_n = \infty$. Then, $\lim_{n\to\infty} b_n^{-1} \sum_{i=1}^{n} b_i x_i = 0$.

Kolmogorov strong law of large numbers

Let $\{X_n : n = 1, 2, ...\}$ be independent and identically distributed (i.i.d.) $\mathbb{R}$–valued random variables, defined on a common probability space $(\Omega, \mathcal{F}, P)$. If $\mathrm{E}(|X_1|) < \infty$, then $n^{-1} \sum_{i=1}^{n} X_i$ converges to $\mathrm{E}(X_1)$ almost surely. Moreover, if $\mathrm{E}(|X_1|) = \infty$, then $n^{-1} \sum_{i=1}^{n} X_i$ diverges a.s.

Proof. For the case where $\mathrm{E}(|X_1|) < \infty$, assume without loss of generality that $\mathrm{E}(X_1) = 0$. The key idea is that to prove $n^{-1} \sum_{i=1}^{n} X_i \to^{\text{a.s.}} 0$ it suffices to prove $n^{-1} \sum_{i=1}^{n} Y_i \to^{\text{a.s.}} 0$, where $Y_n = X_n 1_{\{|X_n| < n\}}$ is a truncated version of X_n. This is based on the following lemmas:

Lemma 1: Let $\{X_n : n = 1, 2, ...\}$ be i.i.d. $\mathbb{R}$–valued random variables on probability space $(\Omega, \mathcal{F}, P)$. Then, $\mathrm{E}(|X_1|) < \infty$ if and only if $P(\limsup_{n\to\infty} \{|X_n| \geq n\}) = 0$.

Lemma 2: Let Y be an $\mathbb{R}^+$–valued random variable on probability space $(\Omega, \mathcal{F}, P)$. Then, $\sum_{n=1}^{\infty} P(Y \geq n) \leq \mathrm{E}(Y) \leq 1 + \sum_{n=1}^{\infty} P(Y \geq n)$.

Almost sure convergence for $n^{-1} \sum_{i=1}^{n} Y_i$ can be established using Theorem 2 (truncation ensures finiteness of $\mathrm{Var}(Y_n)$, even though no assumption is made on finiteness of $\mathrm{Var}(X_n)$).