AMS 261: Probability Theory (Fall 2017)

Limits of sequences of events in a probability space

Limit supremum and limit infimum for real sequences

Consider a sequence of real numbers $\{a_n : n = 1, 2, ...\}$. Although the sequence may not be convergent, we can always define its limit supremum and limit infimum, which are concepts that help describe the asymptotic behavior of the sequence. For each n, let

$$I_n = \inf\{a_m : m \geq n\} \quad \text{and} \quad S_n = \sup\{a_m : m \geq n\}.$$

Note that $\{I_n : n = 1, 2, ...\}$ is an increasing sequence and $\{S_n : n = 1, 2, ...\}$ is a decreasing sequence. Hence, both $\lim_{n\to\infty} I_n$ and $\lim_{n\to\infty} S_n$ exist, where the former may be $+\infty$ and the latter may be $-\infty$. Then the limit infimum and limit supremum of the sequence $\{a_n : n = 1, 2, ...\}$ are defined by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} I_n \quad \text{and} \quad \limsup_{n\to\infty} a_n = \lim_{n\to\infty} S_n.$$

It can be shown that for any convergent subsequence $\{a_{n_k} : k = 1, 2, ...\}$ of $\{a_n : n = 1, 2, ...\}$,

$$\liminf_{n\to\infty} a_n \leq \lim_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n.$$

Moreover, if $\{a_n : n = 1, 2, ...\}$ is a convergent sequence, then $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$. Conversely, if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a$, then the sequence $\{a_n : n = 1, 2, ...\}$ is convergent and $\lim_{n\to\infty} a_n = a$.

[For more details, refer to any book on introductory real analysis, for example, "Real analysis and foundations" by Krantz (first edition in 1991, second edition in 2005).]

Indicator functions

Consider a probability space $(\Omega, \mathcal{F}, P)$. Operations on subsets of Ω can be described in terms of operations on the corresponding indicator functions. For any subset A of Ω, the indicator function, 1_A, is defined for all $\omega \in \Omega$ by $1_A(\omega) = 1$, if $\omega \in A$, and $1_A(\omega) = 0$ otherwise. The following are indicator-function versions of standard set-theoretic operations:

- $1_{A^c} = 1 - 1_A$

- $1_{A \cap B} = 1_A 1_B = \min\{1_A, 1_B\}$

- $1_{A \cup B} = 1_A + 1_B - 1_{A \cap B} = \max\{1_A, 1_B\}$

- $1_{\bigcup_{n=1}^{\infty} A_n} = \max_n\{1_{A_n}\}$

- $1_{\bigcap_{n=1}^{\infty} A_n} = \min_n\{1_{A_n}\}$

Limit infimum, limit supremum, and limit for sequences of sets

Let $\{A_n : n = 1, 2, ...\}$ be a sequence of subsets of Ω (typically, measurable subsets, i.e., $A_n \in \mathcal{F}$). The asymptotic behavior of the sequence $\{A_n : n = 1, 2, ...\}$ can be studied through the sequence of the corresponding indicator functions $\{1_{A_n}(\omega) : n = 1, 2, ...\}$, for any $\omega \in \Omega$. In particular, if $\lim_{n \to \infty} 1_{A_n}(\omega)$ exists for all $\omega \in \Omega$, we will say that the limit of $\{A_n : n = 1, 2, ...\}$ exists. Note that, provided it exists, $\lim_{n \to \infty} 1_{A_n}(\omega) = g(\omega)$ is, for each $\omega \in \Omega$, either 0 or 1, therefore $g(\cdot)$ is the indicator function of some set A in Ω, i.e., $g = 1_A$. We will write $A = \lim_{n \to \infty} A_n$.

In general, $\lim_{n \to \infty} 1_{A_n}(\omega)$ might not exist for some ω, in which case, $\lim_{n \to \infty} A_n$ does not exist. However, $\liminf_{n \to \infty} 1_{A_n}(\omega)$ and $\limsup_{n \to \infty} 1_{A_n}(\omega)$ always exist for any $\omega \in \Omega$. Again, these are indicator functions of sets in Ω, which will be referred to as the *limit infimum* and *limit supremum* of the sequence $\{A_n : n = 1, 2, ...\}$, and denoted by $\liminf_{n \to \infty} A_n$ and $\limsup_{n \to \infty} A_n$, respectively. Based on these definitions and the results for numerical sequences of reals, we have that $\lim_{n \to \infty} A_n$ exists **if and only if** $\liminf_{n \to \infty} A_n = \limsup_{n \to \infty} A_n = A$, and in this case, $A = \lim_{n \to \infty} A_n$.

It can be shown that

$$\liminf_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m \quad \text{and} \quad \limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Hence, if $\omega \in \liminf_{n \to \infty} A_n$, then there exists an n such that $\omega \in A_m$ for all $m \geq n$ (i.e., "$\omega \in A_n$ eventually"), whereas if $\omega \in \limsup_{n \to \infty} A_n$, then for any n, there exists $m \geq n$ such that $\omega \in A_m$ (i.e., "$\omega \in A_n$ infinitely often").

The sets $\liminf_{n \to \infty} A_n$ and $\limsup_{n \to \infty} A_n$ are very useful in the study of the asymptotic behavior for a sequence of events $\{A_n : n = 1, 2, ...\}$ in $(\Omega, \mathcal{F}, P)$. We will prove some of the key results that provide the values of, or bounds for, $P(\liminf_{n \to \infty} A_n)$ and $P(\limsup_{n \to \infty} A_n)$.